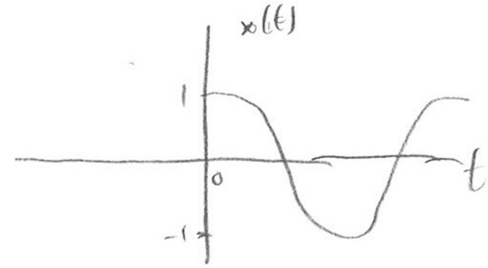
1. $x(t) = \cos(\omega t)\, u(t)$, $h(t) = \sin(\omega t)\, u(t)$

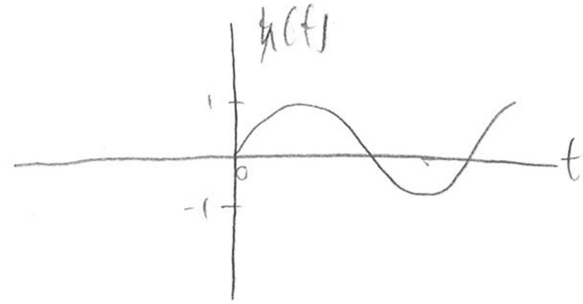


$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

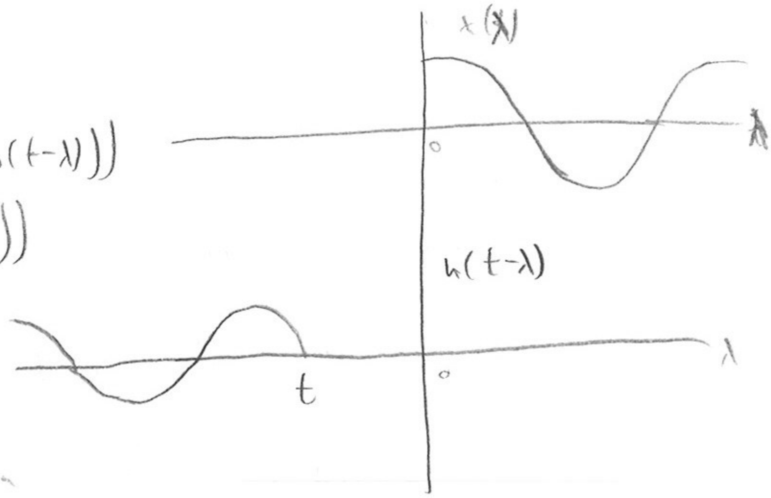
$$= \int_{-\infty}^{\infty} \cos(\omega\lambda)\, u(\lambda) \sin(\omega(t-\lambda))\, u(t-\lambda)\, d\lambda$$

$$= \int_{0}^{\infty} \cos(\omega\lambda) \sin(\omega(t-\lambda))\, u(t-\lambda)\, d\lambda$$

$$= \int_{0}^{t} \cos(\omega\lambda) \sin(\omega(t-\lambda))\, d\lambda$$



manipulate the integrand:



$$\cos(\omega\lambda) \sin(\omega(t-\lambda))$$

$$= \frac{1}{2}\left(\sin(\omega\lambda + \omega(t-\lambda)) - \sin(\omega\lambda - \omega(t-\lambda))\right)$$

$$= \frac{1}{2}\left(\sin(\omega t) - \sin(\omega(2\lambda - t))\right)$$

Or if you're like me, you can't remember trig identities, but you can remember Euler's formula

$$\cos(\omega\lambda) \sin(\omega(t-\lambda)) = \frac{1}{2}\left(e^{j\omega\lambda} + e^{-j\omega\lambda}\right) \frac{1}{2j}\left(e^{j\omega t} e^{-j\omega\lambda} - e^{-j\omega t} e^{+j\omega\lambda}\right)$$

$$= \frac{1}{4j}\left(e^{j\omega t} - e^{-j\omega t} e^{2j\omega\lambda} + e^{j\omega t} e^{-2j\omega\lambda} - e^{-j\omega t}\right)$$

$$= \frac{1}{4j}\left(e^{j\omega t} - e^{-j\omega t} - \left(e^{j\omega(2\lambda - t)} - e^{-j\omega(2\lambda - t)}\right)\right)$$

$$= \frac{1}{2}\sin(\omega t) - \frac{1}{2}\sin(\omega(2\lambda - t))$$

1. cont

$$\int_0^t \frac{1}{2}\sin(\omega t)\,d\lambda - \int_0^t \frac{1}{2}\sin(\omega(2\lambda - t))\,d\lambda \qquad \begin{aligned} v &= 2\lambda - t \\ dv &= 2\,d\lambda \end{aligned}$$

$$\frac{1}{2}\sin(\omega t)\int_0^t d\lambda - \int_{-t}^{2t-t} \frac{1}{4}\sin(\omega v)\,dv$$

$$\frac{t}{2}\sin(\omega t) - \underbrace{\int_{-t}^{t}}_{\text{symmetric}} \frac{1}{4}\underbrace{\sin(\omega v)}_{\text{odd}}\,dv = \frac{t}{2}\sin(\omega t)$$

$$y(t) = \begin{cases} 0 & t \le 0 \\ \frac{t}{2}\sin(\omega t) & t > 0 \end{cases}$$

2)

$$\int_{-\infty}^{\infty} (x * y)(t)\, dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\lambda)\, y(t-\lambda)\, d\lambda \right) dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\lambda)\, y(t-\lambda)\, dt\, d\lambda \qquad \text{(by hypothesis)}$$

$$= \int_{-\infty}^{\infty} x(\lambda) \left(\int_{-\infty}^{\infty} y(t-\lambda)\, dt \right) d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda) \left(\underbrace{\int_{-\infty}^{\infty} y(u)\, du}_{\text{constant}} \right) d\lambda$$

$$= \left(\int_{-\infty}^{\infty} y(u)\, du \right) \left(\int_{-\infty}^{\infty} x(\lambda)\, d\lambda \right)$$

This says the area under the convolution product is the same as the product of the areas under the convolved functions. In particular, if

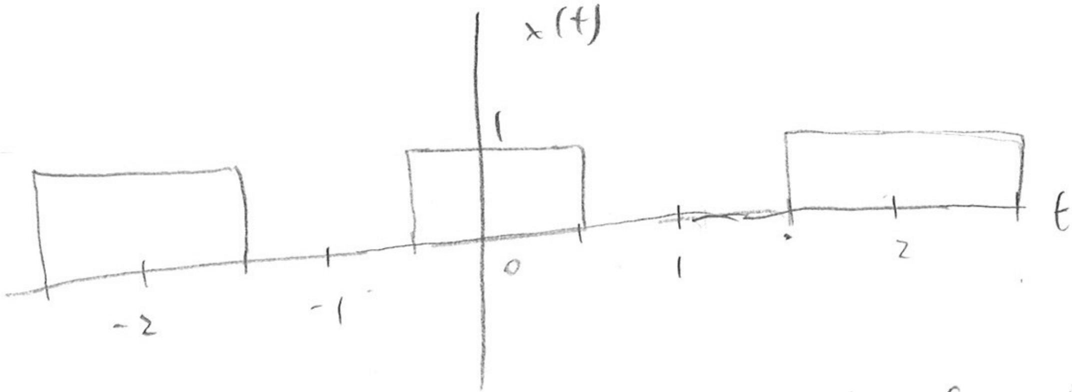
$$\int_{-\infty}^{\infty} x\, dt = \int_{-\infty}^{\infty} y\, dt = 1$$

then $\int_{-\infty}^{\infty} (x * y)\, dt = 1$

This is important because the convolution of two probability density functions is also a pdf. Convolution has a meaningful interpretation in probability theory; $(x * y)$ is the pdf of the random variable $Z = X + Y$ when X has pdf $x(t)$ and Y has pdf $y(t)$ and X and Y are independent.

3.a)

$$x(t) = \Pi(t+2) + \Pi(t) + \Pi(t-2)$$



3.b)

The easy way to do this is with Fourier transforms,

$$\Pi(t) \xrightarrow{\mathcal{F}} \text{sinc}(f)$$

$$\Pi(t \pm 2) \xrightarrow{\mathcal{F}} e^{\pm 2\pi j f 2}\,\text{sinc}(f)$$

$$x(t) \xrightarrow{\mathcal{F}} X(f) = \left(e^{4\pi j f} + 1 + e^{-4\pi j f}\right)\text{sinc}(f)$$

$$Y(f) = X(f)\,X(f) = \left(e^{4\pi j f} + 1 + e^{-4\pi j f}\right)^2 \text{sinc}^2(f)$$

$$= \left(e^{8\pi j f} + e^{4\pi j f} + 1 + e^{4\pi j f} + 1 + e^{-4\pi j f} + 1 + e^{-4\pi j f} + e^{-8\pi j f}\right)\text{sinc}^2(f)$$

$$= \left(e^{8\pi j f} + 2e^{4\pi j f} + 3 + 2e^{-4\pi j f} + e^{-8\pi j f}\right)\text{sinc}^2(f)$$

$$\text{sinc}^2(f) \xrightarrow{\mathcal{F}^{-1}} \Lambda(t) \qquad \Lambda(t) = (\Pi * \Pi)(t) = \begin{cases} 0 & |t| \geq 1 \\ 1-|t| & |t| < 1 \end{cases}$$

$$\text{sinc}^2(f)\,e^{-2\pi j f t_0} \xrightarrow{\mathcal{F}^{-1}} \Lambda(t - t_0)$$

$$Y(f) \xrightarrow{\mathcal{F}^{-1}} \boxed{\Lambda(t+4) + 2\Lambda(t+2) + 3\Lambda(t) + 2\Lambda(t-2) + \Lambda(t-4) = y(t)}$$

3.b cont)

using rules of convolution:

$$x * x = \left(\Pi(t+2) + \Pi(t) + \Pi(t-2)\right) * \left(\Pi(t+2) + \Pi(t) + \Pi(t-2)\right)$$

$$= \Pi(t+2) * \Pi(t+2) + \Pi(t+2) * \Pi(t) + \Pi(t+2) * \Pi(t-2)$$
$$+ \Pi(t) * \Pi(t+2) + \Pi(t) * \Pi(t) + \Pi(t) * \Pi(t-2)$$
$$+ \Pi(t-2) * \Pi(t+2) + \Pi(t) * \Pi(t-2) + \Pi(t-2) * \Pi(t-2)$$

(linearity)

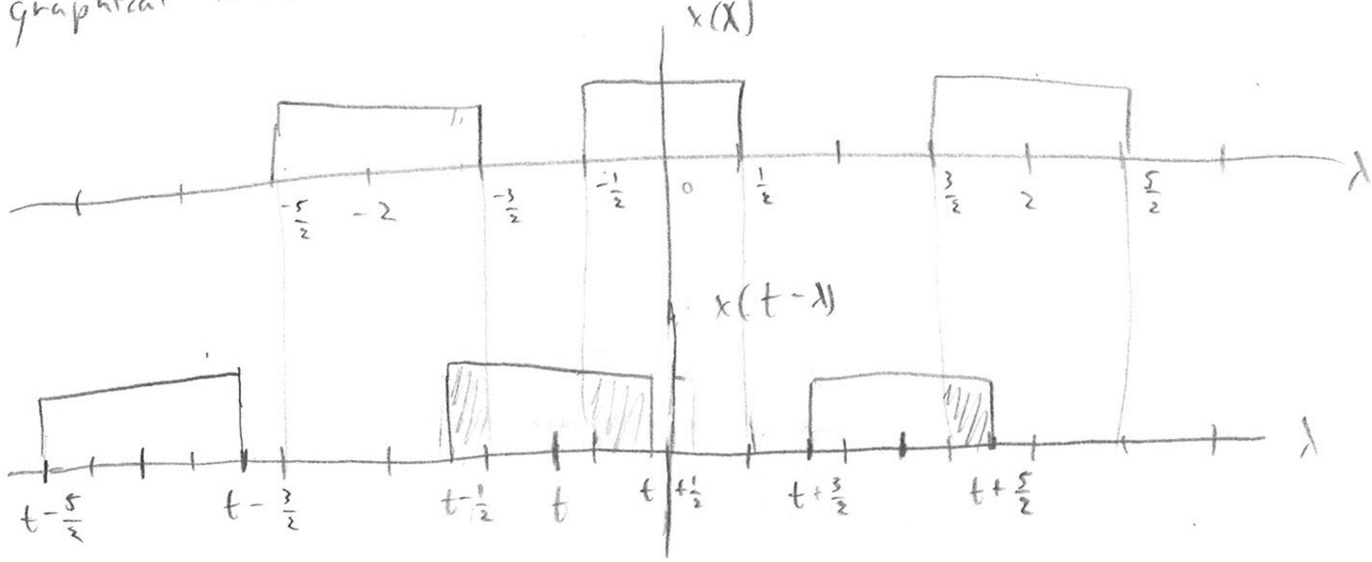
$$= \Lambda(t+4) + \Lambda(t+2) + \Lambda(t)$$
$$+ \Lambda(t+2) + \Lambda(t) + \Lambda(t-2)$$
$$+ \Lambda(t) + \Lambda(t-2) + \Lambda(t-4)$$

(translation invariance and $\Pi * \Pi = \Lambda$)

$$= \boxed{\Lambda(t+4) + 2\Lambda(t+2) + 3\Lambda(t) + 2\Lambda(t-2) + \Lambda(t-4)}$$

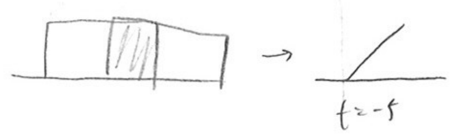
3.b cont)

graphical solution

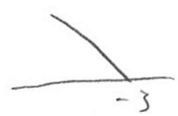


$t+\frac{5}{2} < -\frac{5}{2} \rightarrow t < -5$: $y = 0$

$t+\frac{5}{2} > -\frac{5}{2}$ and $t+\frac{3}{2} < -\frac{5}{2} \rightarrow t > -5$ and $t < -4$; $y = t-5$



$t+\frac{3}{2} > -\frac{5}{2}$, $t+\frac{3}{2} < -\frac{3}{2} \rightarrow -4 < t < -3$; $y = -(t-3)$

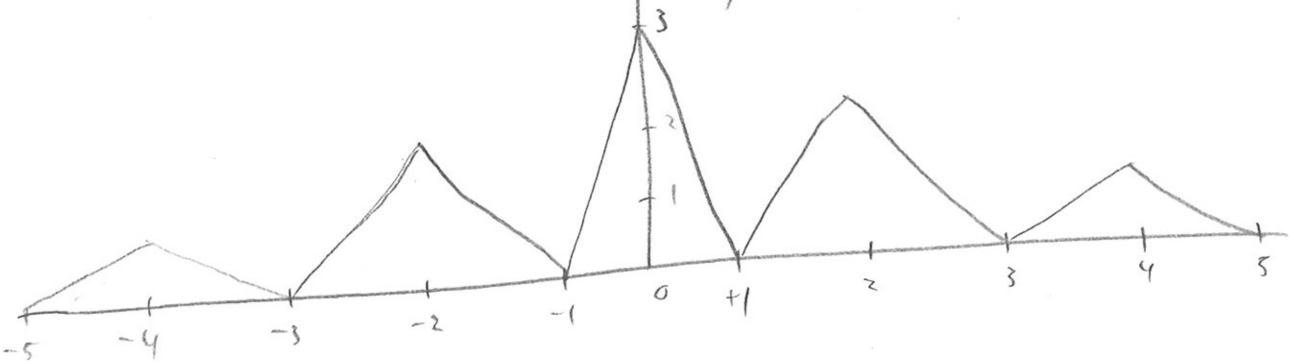


and so on...

$$y(t) = \begin{cases} 0 & t < -5 \\ t-5 & -5 < t < -4 \\ -(t-3) & -4 < t < -3 \\ 2(t-3) & -3 < t < -2 \\ -2(t-1) & -2 < t < -1 \\ 3(t-1) & -1 < t < 0 \\ y(-t) & t > 0 \end{cases}$$

3.c

$y(t) = (x * x)(t)$



4.a.

$$x_1 * (x_2 * x_3) = x_1 * \left(\overbrace{\int_{-\infty}^{\infty} x_2(t-\lambda)\, x_3(\lambda)\, d\lambda}^{g(t)} \right)$$

$$= \int_{-\infty}^{\infty} \underset{\uparrow \text{ const w.r.t. } \lambda}{x_1(\tau)} \left(\underbrace{\int_{-\infty}^{\infty} x_2((t-\tau)-\lambda)\, x_3(\lambda)\, d\lambda}_{g(t-\tau)} \right) d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau-\lambda)\, x_3(\lambda)\, d\lambda\, d\tau$$

$$(x_1 * x_2) * x_3 = \left(\overbrace{\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau}^{h(t)} \right) * x_3(t)$$

$$= \int_{-\infty}^{\infty} \left(\underbrace{\int_{-\infty}^{\infty} x_1(\tau)\, x_2((t-\lambda)-\tau)\, d\tau}_{h(t-\lambda)} \right) \underset{\uparrow \text{ const w.r.t. } \tau}{x_3(\lambda)}\, d\lambda$$

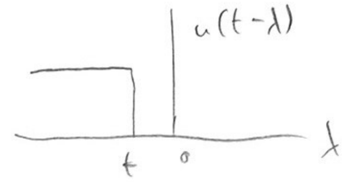
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\lambda-\tau)\, x_3(\lambda)\, d\tau\, d\lambda$$

The integrand is the same in both cases, and by hypothesis we can change the order of integration, so $x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$ (convolution is associative). Fubini's theorem gives the conditions when this is a valid assumption.

4.b)

$$x_1(t) = u(t), \quad x_2(t) = \frac{d}{dt}e^{-t^2} = -2te^{-t^2}, \quad x_3(t) = u(-t)$$

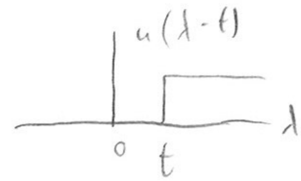
$$x_1 * x_2 = \int_{-\infty}^{\infty} -2\lambda e^{-\lambda^2} u(t-\lambda)\,d\lambda = \int_{-\infty}^{t} -2\lambda e^{-\lambda^2}\,d\lambda$$



$$= e^{-\lambda^2}\Big]_{\lambda=-\infty}^{t} = e^{-t^2} - 0$$

$u(-(t-\lambda))$

$$(x_1 * x_2) * x_3 = \int_{-\infty}^{\infty} e^{-\lambda^2} u(\lambda - t)\,d\lambda$$



$$= \int_{t}^{\infty} e^{-\lambda^2}\,d\lambda = \frac{\sqrt{\pi}}{2}\,\mathrm{erfc}(t)$$

complementary error function

$$x_2 * x_3 = \int_{-\infty}^{\infty} -2\lambda e^{-\lambda^2} u(\lambda - t)\,d\lambda$$

$$= \int_{t}^{\infty} -2\lambda e^{-\lambda^2}\,d\lambda = e^{-\lambda^2}\Big]_{\lambda=t}^{\infty} = 0 - e^{-t^2}$$

$$x_1 * (x_2 * x_3) = \int_{-\infty}^{\infty} -e^{-\lambda^2} u(t-\lambda)\,d\lambda$$

$$= \int_{-\infty}^{t} -e^{-\lambda^2}\,d\lambda \qquad v = -\lambda$$

$$dv = -d\lambda$$

$$= \int_{\infty}^{-t} -e^{-v^2}\,(-dv) = -\int_{-t}^{\infty} e^{-v^2}\,dv = -\frac{\sqrt{\pi}}{2}\,\mathrm{erfc}(-t)$$

these are not equal!

$$(x_1 * x_2) * x_3 - x_1 * (x_2 * x_3) = \int_{t}^{\infty} e^{-\lambda^2}\,d\lambda - \int_{-\infty}^{t} -e^{-\lambda^2}\,d\lambda$$

$$= \int_{t}^{\infty} e^{-\lambda^2}\,d\lambda + \int_{-\infty}^{t} e^{-\lambda^2}\,d\lambda$$

$$= \int_{-\infty}^{\infty} e^{-\lambda^2}\,d\lambda = \sqrt{\pi} \neq 0$$

You don't need to know that $\int_{-\infty}^{\infty} e^{-\lambda^2}\,d\lambda = \sqrt{\pi}$, it is obviously non zero because $e^{-\lambda^2} > 0 \;\; \forall \lambda \in \mathbb{R}$